Covariant Treatment of Linearized Magneto-Fluid Dynamics*

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A perfectly conducting, nonviscous fluid is assumed to be permeated by an inhomogeneous, time independent magnetic field satisfying the magnetohydrostatic equation.

This fluid shall be perturbed by hydromagnetic waves of small amplitude. The wave equation is expressed in a completely covariant form. This formulation allows to search for simplifying coordinate systems intrinsically adapted to the undisturbed field configuration. Hydromagnetic waves with the velocity vector satisfying the equation div $\boldsymbol{v}=0$, which are the natural generalization of ALFVÉN waves have been investigated especially. It is shown that there are simplifying coordinate systems for the hydromagnetic wave equation. Especially it is shown that the solution of the ALFVÉN wave equation can always be reduced to the solution of an ordinary differential equation if in the equilibrium configuration the current is either perpendicular on the magnetic field line or zero.

In linearized magneto-fluid dynamics one considers the propagation of a disturbance or wave of infinitesimal amplitude in a liquid conductor permeated by a time independent magnetic field. In the special case where this time independent magnetic field is uniform, the propagation of the hydromagnetic waves is known in every detail. In particular it is known that there exist a type of hydromagnetic wave, transversal in nature known as an ALFVÉN wave. ALFVÉN waves satisfy the additional condition that the velocity vector shall satisfy the equation div $\boldsymbol{v}=0$.

ALFVÉN waves for this reason can propagate in a compressible fluid as well as in an incompressible fluid.

The usual treatment of the hydromagnetic wave propagation is restricted to the case where the magnetic field lines of the undisturbed field are straight and uniform. In general, however, the magnetic field lines of an undisturbed field are neither straight nor uniform but only of such a character as to satisfy the magnetohydrostatic equation.

In a previous paper [1] the propagation of ALFVÉN waves in an external, but otherwise in general, inhomogeneous magnetic field has been considered. In such a case it is always possible to derive the magnetic field as a gradient of a scalar potential satisfying LAPLACE's equation. Within the family of field lines there exist then always a family of equipotential surfaces. The magnetic field lines intersect these surfaces everywhere perpendicularly. For such a field which is free of volume-currents the magnetohydrostatic equation is satisfied automatically.

A curvilinear coordinate system was introduced in which the magnetic field lines were directed along one family of coordinate lines. The two other families of coordinate lines were placed into the equipotential surfaces. The hydromagnetic wave equation written down in such a coordinate system took a much simpler form. Using this coordinate system, it was shown that the solution for the propagation of an ALFVÉN wave could be reduced to the solution of an ordinary differential equation if the field lines had no torsion.

In the present investigation we would like to consider the most general case of linearized magneto-fluid dynamics permitting the existence of volume currents in the undisturbed field configuration.

In contrast to the case where no volume currents are present, there does not exist, in general a family of surfaces which is intersected by the field lines perpendicularly everywhere. It is then, in general, not possible to set up a coordinate system as in the case of a current-free magnetic field, where two of the coordinate lines could be chosen as being located in the equipotential surfaces.

It is, however, possible to use as coordinate surfaces, the so-called magnetic surfaces [2] in which the magnetic field lines and the electric current lines are located. The two coordinate lines in this surface are

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[1] R. GAJEWSKI and F. WINTERBERG, Boeing Scientific Research Report DI-82-0111; Ann. Phys., New York, to be published.

[2] M. D. KRUSKAL and R. M. KULSRUD, Phys. Fluids 1, 265 [1958].

properly directed along the field and electric current lines. The third coordinate line can be directed perpendicularly to both of it.

This choice for the coordinate lines fails in the case of a force-free magnetic field, where another consideration to obtain a simplifying coordinate system will be exploited.

1. The Hydromagnetic Wave Equation

We start with the well known equations of magneto-fluid dynamics. These equations are EULER's equation

$$\varrho \frac{d\boldsymbol{v}}{dt} = \frac{1}{c} \boldsymbol{j} \times \boldsymbol{H} - \operatorname{grad} \varrho\,, \tag{1.1}$$

OHM's law for infinite conductivity

$$\boldsymbol{E} + (1/c)\, \boldsymbol{v} \times \boldsymbol{H} = 0\,, \tag{1.2}$$

MAXWELL's equations

$$\partial \boldsymbol{H}/\partial t = -c \operatorname{curl} \boldsymbol{E}\,, \tag{1.3}$$

$$(4\pi/c)\, \boldsymbol{j} = \operatorname{curl} \boldsymbol{H}\,, \tag{1.4}$$

$$\operatorname{div} \boldsymbol{H} = 0\,, \tag{1.5}$$

Equation of continuity

$$\partial \varrho/\partial t + \operatorname{div} \varrho\, \boldsymbol{v} = 0\,. \tag{1.6}$$

Finally we assume a baratropic equation of state:

$$\varrho = \varrho(p) \tag{1.7}$$

($\boldsymbol{v}$ fluid velocity, $\boldsymbol{j}$ current density, c velocity of light, $\boldsymbol{H}$ magnetic field, ϱ density, p pressure, $\boldsymbol{E}$ electric field).

First we combine (1.1) and (1.4):

$$\varrho \frac{d\boldsymbol{v}}{dt} = -\frac{1}{4\pi} \boldsymbol{H} \times \operatorname{curl} \boldsymbol{H} - \operatorname{grad} p\,. \tag{1.8}$$

Combination of (1.2) and (1.3) yields:

$$\partial \boldsymbol{H}/\partial t = \operatorname{curl} \boldsymbol{v} \times \boldsymbol{H}\,. \tag{1.9}$$

In order to carry out the linearization we expand

$$\boldsymbol{H} = \boldsymbol{H}_0(\boldsymbol{r}) + \boldsymbol{h}(\boldsymbol{r}, t)\,, \tag{1.10}$$

$$p = p_0(\boldsymbol{r}) + p'(\boldsymbol{r}, t)\,, \tag{1.11}$$

$$\varrho = \varrho_0(\boldsymbol{r}) + \varrho'(\boldsymbol{r}, t)\,. \tag{1.12}$$

We consider $\boldsymbol{v}$, p' and ϱ' small of the same order as $\boldsymbol{h}$ and neglect terms higher than the first order in $\boldsymbol{h}$, $\boldsymbol{v}$, p' and ϱ'. We obtain then instead of (1.8):

$$\varrho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{4\pi} [\boldsymbol{H}_0 \times \operatorname{curl} \boldsymbol{h} + \boldsymbol{h} \times \operatorname{curl} \boldsymbol{H}_0] - \frac{1}{4\pi} [\boldsymbol{H}_0 \times \operatorname{curl} \boldsymbol{H}_0] - \operatorname{grad} p_0 - \operatorname{grad} p'. \tag{1.13}$$

Now we observe that the undisturbed equilibrium must be satisfied by the magnetohydrostatic equation. This equation is:

$$(1/4\pi)\, \boldsymbol{H}_0 \times \operatorname{curl} \boldsymbol{H}_0 = -\operatorname{grad} p_0\,. \tag{1.14}$$

This simplifies (1.13) to

$$\varrho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{4\pi} [\boldsymbol{H}_0 \times \operatorname{curl} \boldsymbol{h} + \boldsymbol{h} \times \operatorname{curl} \boldsymbol{H}_0] - \operatorname{grad} p'. \tag{1.15}$$

Linearization of (1.9) yields

$$\partial \boldsymbol{h}/\partial t = \operatorname{curl} \boldsymbol{v} \times \boldsymbol{H}_0\,. \tag{1.16}$$

Equation (1.5) splits up into

$$\operatorname{div} \boldsymbol{H}_0 = 0\,, \qquad \operatorname{div} \boldsymbol{h} = 0 \tag{1.17}$$

and finally linearizing equation (1.6) yields

$$\frac{\partial \varrho'}{\partial t} + \varrho_0 \operatorname{div} \boldsymbol{v} + (\boldsymbol{v} \cdot \operatorname{grad} \varrho_0) = 0$$

or
$$\frac{\partial \varrho'}{\partial t} + \varrho_0 \operatorname{div} \boldsymbol{v} + \frac{\partial \varrho}{\partial p} (\boldsymbol{v} \cdot \operatorname{grad} p_0) = 0\,. \tag{1.18}$$

Now we observe that

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial \varrho} \frac{\partial \varrho}{\partial t} \simeq \frac{\partial p}{\partial \varrho} \frac{\partial \varrho'}{\partial t}\,. \tag{1.19}$$

$$\partial p/\partial \varrho = a^2 \tag{1.20}$$

is the square of the sound velocity, which shall be assumed constant. This is for example the case if the fluid satisfies the gas equation and has constant temperature.

Combining (1.18) and (1.19) results in

$$\frac{\partial p'}{\partial t} = -\varrho_0 a^2 \operatorname{div} \boldsymbol{v} - (\boldsymbol{v} \cdot \operatorname{grad} p_0)\,. \tag{1.21}$$

After differentiating (1.15) and (1.16) with respect to time, $\boldsymbol{h}$ can be eliminated from both equations. p' furthermore can be eliminated by (1.21) and p_0 from (1.14). The result is:

$$\varrho_0 \frac{\partial^2 \boldsymbol{v}}{\partial t^2} = -\frac{1}{4\pi} [\boldsymbol{H}_0 \times \operatorname{curl} \operatorname{curl} \boldsymbol{v} \times \boldsymbol{H}_0 + [\operatorname{curl} \boldsymbol{v} \times \boldsymbol{H}_0] \times \operatorname{curl} \boldsymbol{H}_0] + \varrho_0 a^2 \operatorname{grad} \operatorname{div} \boldsymbol{v} - \frac{1}{4\pi} [\boldsymbol{H}_0 \times \operatorname{curl} \boldsymbol{H}_0] \operatorname{div} \boldsymbol{v} - \frac{1}{4\pi} \operatorname{grad} (\boldsymbol{v} \cdot [\boldsymbol{H}_0 \times \operatorname{curl} \boldsymbol{H}_0])\,. \tag{1.22}$$

We introduce instead of $\boldsymbol{H}_0$ the vector $\boldsymbol{u}$ defined by

$$\boldsymbol{u} = \boldsymbol{H}_0/\sqrt{4\pi} \tag{1.23}$$

and assume periodic time dependence of the form:

$$\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{v}(\boldsymbol{r})\, e^{-i\omega t} \tag{1.24}$$

and similar for $\varrho'(\boldsymbol{r}, t)$. The result upon (1.24) is:

$$\begin{aligned}\varrho_0\, \omega^2\, \boldsymbol{v} = {} & \boldsymbol{u} \times \operatorname{curl}\operatorname{curl} \boldsymbol{v} \times \boldsymbol{u} + [\operatorname{curl} \boldsymbol{v} \times \boldsymbol{u}] \times \operatorname{curl} \boldsymbol{u} - \varrho_0\, a^2 \operatorname{grad}\operatorname{div} \boldsymbol{v} \\ & + [\boldsymbol{u} \times \operatorname{curl} \boldsymbol{u}] \operatorname{div} \boldsymbol{v} + \operatorname{grad}(\boldsymbol{v} \cdot [\boldsymbol{u} \times \operatorname{curl} \boldsymbol{u}]) .\end{aligned} \tag{1.25}$$

In addition we have

$$\operatorname{div} \boldsymbol{u} = 0 \tag{1.26}$$

and the equation of continuity:

$$-i\,\omega\,\varrho' + \varrho_0 \operatorname{div} \boldsymbol{v} - a^{-2}(\boldsymbol{v} \cdot [\boldsymbol{u} \times \operatorname{curl} \boldsymbol{u}]) = 0 . \tag{1.27}$$

Equations (1.25) is the well known general magnetohydrodynamic wave equation [3].

2. Covariant Form of the Magnetohydrodynamic Wave Equation

We consider a general curvilinear coordinate system given by the line element

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta , \tag{2.1}$$

$g_{\alpha\beta}$ is the metric tensor of the curvilinear coordinate system.

In such a general curvilinear coordinate system equation (1.27) can be written as follows:

$$\begin{aligned}\varrho_0\, \omega^2\, v^\sigma = {} & \varepsilon^{\sigma\alpha\pi}\, \varepsilon_{\pi\varrho\mu}\, \varepsilon^{\mu\lambda\tau}\, \varepsilon_{\tau\beta\gamma}\, u_\alpha (v^\beta u^\gamma)_{;\lambda}{}^{;\varrho} + \varepsilon^{\sigma\mu\alpha}\, \varepsilon_{\alpha\pi\varrho}\, \varepsilon^{\tau\beta\gamma}\, \varepsilon_{\mu\lambda\tau} (v_\beta\, u_\gamma)^{;\lambda}\, u^{\varrho;\pi} \\ & - \varrho_0\, a^2 (v^\gamma{}_{;\gamma})^{;\sigma} + \varepsilon^{\sigma\tau\lambda}\, \varepsilon_{\lambda\nu\pi}\, u_\tau\, u^{\pi;\nu}\, v^\beta{}_{;\beta} + \varepsilon^{\alpha\beta\gamma}\, \varepsilon_{\mu\varrho\alpha} (v^\mu\, u^\varrho\, u_{\gamma;\beta})^{;\sigma} .\end{aligned} \tag{2.2}$$

For (1.28) we write $\quad u^\beta{}_{;\beta} = 0 \quad$ or $\quad u_\gamma{}^{;\gamma} = 0 . \quad$ (2.3a), (2.3b)

The equation of continuity in covariant form is:

$$-i\,\omega\,\varrho' + \varrho_0\, v^\beta{}_{;\beta} - a^{-2}\, \varepsilon^{\alpha\beta\gamma}\, \varepsilon_{\mu\varrho\alpha}\, v^\mu\, u^\varrho\, u_{\gamma;\beta} = 0 \quad \text{or} \quad -i\,\omega\,\varrho' + \varrho_0\, v_\gamma{}^{;\gamma} - a^{-2}\, \varepsilon_{\alpha\beta\gamma}\, \varepsilon^{\mu\varrho\alpha}\, v_\mu\, u_\varrho\, u^{\gamma;\beta} = 0. \tag{2.4a, b}$$

Semicolons define covariant and commas ordinary derivatives. The customary summation and index conventions have been used.

The symbols $\varepsilon_{\alpha\beta\gamma}$, $\varepsilon^{\alpha\beta\gamma}$ are the completely antisymmetric RICCI-symbols, being equal to 1 for an even and -1 for an odd permutation of $\alpha\beta\gamma$, in all other cases zero. Use was further made of the well known formulas for the covariant expressions of a vector product $\boldsymbol{A} \times \boldsymbol{B}$

$$g^{1/2}\, \varepsilon_{\alpha\beta\gamma}\, A^\beta\, B^\gamma \quad \text{or} \quad g^{-1/2}\, \varepsilon^{\alpha\beta\gamma}\, A_\beta\, B_\gamma$$

and for the curl $\boldsymbol{A}$

$$g^{1/2}\, \varepsilon_{\alpha\beta\gamma}\, A^{\gamma;\beta} \quad \text{or} \quad g^{-1/2}\, \varepsilon^{\alpha\beta\gamma}\, A_{\gamma;\beta} .$$

With the identities

$$\varepsilon^{\mu\lambda\tau}\, \varepsilon_{\tau\beta\gamma} = \delta_\beta{}^\mu\, \delta_\gamma{}^\lambda - \delta_\gamma{}^\mu\, \delta_\beta{}^\lambda \tag{2.5}$$

equation (2.2) can be brought into the form:

$$\begin{aligned}\varrho_0\, \omega^2\, v^\sigma = {} & u_\alpha [(u^\sigma v^\beta)_{;\beta}{}^{;\alpha} - (u^\alpha v^\beta)_{;\beta}{}^{;\alpha} + (u^\beta v^\alpha{}_{;\beta})^{;\alpha} - (u^\beta v^\sigma{}_{;\beta})^{;\alpha}] + v_\beta{}^{;\gamma}\, u_\gamma [u^{\beta;\sigma} - u^{\sigma;\beta}] \\ & + (v_\beta\, u_\gamma)^{;\beta} [u^{\sigma;\gamma} - u^{\gamma;\sigma}] - \varrho_0\, a^2 (v^\gamma{}_{;\gamma})^{;\sigma} + u_\tau (u^{\tau;\sigma} - u^{\sigma;\tau})\, v^\beta{}_{;\beta} + [v^\mu\, u^\gamma (u_{\gamma;\mu} - u_{\mu;\gamma})]^{;\sigma} .\end{aligned} \tag{2.6a}$$

Instead of (2.6a) we can write by making use of (2.3a, b):

$$\begin{aligned}\varrho_0\, \omega^2\, v^\sigma = {} & u^\alpha\, g^{\sigma\varrho} [(u_\varrho\, v^\beta)_{;\beta\alpha} - (u_\alpha\, v^\beta)_{;\beta\varrho} + (u^\beta\, v_{\alpha;\beta})_{;\varrho} - (u^\beta\, v_{\varrho;\beta})_{;\alpha}] \\ & + g^{\sigma\varrho} [(u_{\beta;\varrho} - u_{\varrho;\beta}) (v^\beta{}_{;\gamma}\, u^\gamma - (v^\gamma\, u^\beta)_{;\gamma}) - \varrho_0\, a^2 (v^\gamma{}_{;\gamma})_{;\varrho} \\ & + u^\tau (u_{\tau;\varrho} - u_{\varrho;\tau})\, v^\beta{}_{;\beta} + (v^\mu\, u^\gamma (u_{\gamma;\mu} - u_{\mu;\gamma}))_{;\varrho}] .\end{aligned} \tag{2.6b}$$

[3] For instance: K. HAIN, R. LÜST and A. SCHLÜTER, Z. Naturforschg. **12 a**, 833 [1957].

For this we can also write:

$$\varrho_0\,\omega^2\,v^\sigma = u^\alpha\,g^{\sigma\varrho}[\,(u_{\varrho;\beta}\,v^\beta)_{;\alpha} - (u_{\alpha;\beta}\,v^\beta)_{;\varrho} - (u^\beta\,v_{\varrho;\beta})_{;\alpha} + (u^\beta\,v_{\alpha;\beta})_{;\varrho} + (u_\varrho\,v^\beta{}_{;\beta})_{;\alpha} - (u_\alpha\,v^\beta{}_{;\beta})_{;\varrho}]$$
$$+\,g^{\sigma\varrho}[\,(u_{\beta;\varrho} - u_{\varrho;\beta})\,(v^\beta{}_{;\gamma}\,u^\gamma - v^\gamma\,u^\beta{}_{;\gamma} - v^\gamma{}_{;\gamma}\,u^\beta) - \varrho_0\,a^2\,(v^\gamma{}_{;\gamma})_{;\varrho} \tag{2.6c}$$
$$+\,u^\tau\,(u_{\tau;\varrho} - u_{\varrho;\tau})\,v^\beta{}_{;\beta} + (v^\mu\,u^\gamma\,(u_{\gamma;\mu} - u_{\mu;\gamma}))_{;\varrho}]$$

or

$$\varrho_0\,\omega^2\,v^\sigma = u^\alpha\,g^{\sigma\varrho}[\,(u_{\varrho;\beta}\,v^\beta - u^\beta\,v_{\varrho;\beta})_{;\alpha} - (u_{\alpha;\beta}\,v^\beta - u^\beta\,v_{\alpha;\beta})_{;\varrho} + (u_\varrho\,v^\beta{}_{;\beta})_{;\alpha} - (u_\alpha\,v^\beta{}_{;\beta})_{;\varrho}]$$
$$+\,g^{\sigma\varrho}[\,(u_{\beta;\varrho} - u_{\varrho;\beta})\,(v^\beta{}_{;\gamma}\,u^\gamma - v^\gamma\,u^\beta{}_{;\gamma} - v^\gamma{}_{;\gamma}\,u^\beta) - \varrho_0\,a^2\,(v^\gamma{}_{;\gamma})_{;\varrho} \tag{2.6d}$$
$$+\,u^\tau\,(u_{\tau;\varrho} - u_{\varrho;\tau})\,v^\beta{}_{;\beta} + (v^\mu\,u^\gamma\,(u_{\gamma;\mu} - u_{\mu;\gamma}))_{;\varrho}]\,.$$

Using the well known relation for the covariant derivative of a vector $\boldsymbol{A}$

$$A_{i;k} - A_{k;i} = A_{i,k} - A_{k,i} \tag{2.7a}$$

and for a scalar S

$$S_{;k} = S_{,k} \tag{2.7b}$$

one can write instead of (2.6d):

$$\varrho_0\,\omega^2\,v^\sigma = u^\alpha\,g^{\sigma\varrho}[\,(u_{\varrho;\beta}\,v^\beta - u^\beta\,v_{\varrho;\beta})_{,\alpha} - (u_{\alpha;\beta}\,v^\beta - u^\beta\,v_{\alpha;\beta})_{,\varrho} + (u_\varrho\,v^\beta{}_{;\beta})_{,\alpha} - (u_\alpha\,v^\beta{}_{;\beta})_{,\varrho}]$$
$$+\,g^{\sigma\varrho}[\,(u_{\beta,\varrho} - u_{\varrho,\beta})\,(v^\beta{}_{;\gamma}\,u^\gamma - v^\gamma\,u^\beta{}_{;\gamma} - v^\gamma{}_{;\gamma}\,u^\beta) - \varrho_0\,a^2\,(v^\gamma{}_{;\gamma})_{,\varrho} \tag{2.6e}$$
$$+\,u^\tau\,(u_{\tau,\varrho} - u_{\varrho,\tau})\,v^\beta{}_{;\beta} + (v^\mu\,u^\gamma\,(u_{\gamma,\mu} - u_{\mu,\gamma}))_{,\varrho}]\,.$$

Still another form for the last equation is:

$$\varrho_0\,\omega^2\,v^\sigma = u^\alpha\,g^{\sigma\varrho}[\,\{g_{\varrho\mu}\,(u^\mu{}_{;\beta}\,v^\beta - u^\beta\,v^\mu{}_{;\beta})\}_{,\alpha} - \{g_{\alpha\mu}\,(u^\mu{}_{;\beta}\,v^\beta - u^\beta\,v^\mu{}_{;\beta})\}_{,\varrho} + (u_\varrho\,v^\beta{}_{;\beta})_{,\alpha} - (u_\alpha\,v^\beta{}_{;\beta})_{,\varrho}]$$
$$+\,g^{\sigma\varrho}[\,(u_{\beta,\varrho} - u_{\varrho,\beta})\,(v^\beta{}_{;\gamma}\,u^\gamma - v^\gamma\,u^\beta{}_{;\gamma} - v^\gamma{}_{;\gamma}\,u^\beta) - \varrho_0\,a^2\,(v^\gamma{}_{;\gamma})_{,\varrho} \tag{2.6f}$$
$$+\,u^\tau\,(u_{\tau,\varrho} - u_{\varrho,\tau})\,v^\beta{}_{;\beta} + (v^\mu\,u^\gamma\,(u_{\gamma,\mu} - u_{\mu,\gamma}))_{,\varrho}]\,.$$

Now since

$$u^\mu{}_{;\beta}\,v^\beta = u^\mu{}_{,\beta}\,v^\beta + \Gamma^\mu_{\beta\tau}\,u^\tau\,v^\beta\,, \qquad u_\beta\,v^\mu{}_{;\beta} = u^\beta\,v^\mu{}_{,\beta} + \Gamma^\mu_{\beta\tau}\,v^\tau\,u^\beta$$

it follows that

$$u^\mu{}_{;\beta}\,v^\beta - u^\beta\,v^\mu{}_{;\beta} = u^\mu{}_{,\beta}\,v^\beta - u^\beta\,v^\mu{}_{,\beta} \tag{2.8a}$$

and similar

$$v^\beta{}_{;\gamma}\,u^\gamma - v^\gamma\,u^\beta{}_{;\gamma} = v^\beta{}_{,\gamma}\,u^\gamma - v^\gamma\,u^\beta{}_{,\gamma}\,. \tag{2.8b}$$

Furthermore

$$(u_\varrho\,v^\beta{}_{;\beta})_{,\alpha} - (u_\alpha\,v^\beta{}_{;\beta})_{,\varrho} = v^\beta{}_{;\beta}\,(u_{\varrho,\alpha} - u_{\alpha,\varrho}) + u_\varrho\,(v^\beta{}_{;\beta})_{,\alpha} - u_\alpha\,(v^\beta{}_{;\beta})_{,\varrho} \tag{2.8c}$$

and

$$v^\beta{}_{;\beta} = g^{-1/2}\,(g^{1/2}\,v^\beta)_{,\beta}\,. \tag{2.8d}$$

With the help of (2.8a), (2.8b), (2.8c) and (2.8d) we can further simplify (2.6f). The result is:

$$\varrho_0\,\omega^2\,v^\sigma = u^\alpha\,g^{\sigma\varrho}[\,\{g_{\varrho\mu}\,(u^\mu{}_{,\beta}\,v^\beta - u^\beta\,v^\mu{}_{,\beta})\}_{,\alpha} - \{g_{\alpha\mu}\,(u^\mu{}_{,\beta}\,v^\beta - u^\beta\,v^\mu{}_{,\beta})\}_{,\varrho}$$
$$+\,g^{-1/2}(g^{1/2}\,v^\beta)_{,\beta}\,(u_{\varrho,\alpha} - u_{\alpha,\varrho}) + u_\varrho\{g^{-1/2}(g^{1/2}\,v^\beta)_{,\beta}\}_{,\alpha} - u_\alpha\{g^{-1/2}(g^{1/2}\,v^\beta)_{,\beta}\}_{,\varrho}]$$
$$+\,g^{\sigma\varrho}[\,(u_{\beta,\varrho} - u_{\varrho,\beta})\,(v^\beta{}_{,\gamma}\,u^\gamma - v^\gamma\,u^\beta{}_{,\gamma}) - g^{-1/2}(g^{1/2}\,v^\gamma)_{,\gamma}\,u^\beta \tag{2.6g}$$
$$-\,\varrho_0\,a^2\{g^{-1/2}(g^{1/2}\,v^\gamma)_{,\gamma}\}_{,\varrho} + u^\tau\,(u_{\tau,\varrho} - u_{\varrho,\tau})\,g^{-1/2}(g^{1/2}\,v^\beta)_{,\beta} + (v^\mu\,u^\gamma\,(u_{\gamma,\mu} - u_{\mu,\gamma}))_{,\varrho}]\,.$$

A further simplification of equation (2.6g) is possible for motions obeying the additional condition $\operatorname{div}\boldsymbol{v} = 0$. Such motions are just what we called the generalization of ALFVÉN waves.

In this case we obtain instead of (2.6g):

$$\varrho_0\,\omega^2\,v^\sigma = u^\alpha\,g^{\sigma\varrho}[\,\{g_{\varrho\mu}\,(u^\mu{}_{,\beta}\,v^\beta - u^\beta\,v^\mu{}_{,\beta})\}_{,\alpha} - \{g_{\alpha\mu}\,(u^\mu{}_{,\beta}\,v^\beta - u^\beta\,v^\mu{}_{,\beta})\}_{,\varrho}]$$
$$+\,g^{\sigma\varrho}[\,(u_{\beta,\varrho} - u_{\varrho,\beta})\,(v^\beta{}_{,\gamma}\,u^\gamma - v^\gamma\,u^\beta{}_{,\gamma}) + (v^\mu\,u^\gamma\,(u_{\gamma,\mu} - u_{\mu,\gamma}))_{,\varrho}]\,. \tag{2.9a}$$

Equation (2.9a) must be supplemented by

$$(g^{1/2}\,v^\beta)_{,\beta} = 0\,. \tag{2.9b}$$

Finally the equation of continuity (2.4a) can be written by using (2.5) as follows:

$$-i\,\omega\,\varrho' + \varrho_0\,g^{-1/2}(g^{1/2}\,v^\beta)_{,\beta} - a^{-2}[v^\mu\,u^\gamma\,(u_{\gamma,\mu} - u_{\mu,\gamma})] = 0\,. \tag{2.10}$$

3. The Coordinate System

The advantage of the covariant formulation is that the coordinate system must not be specified. It is, therefore, obvious to search for coordinate systems in which the equations (2.6g), or (2.9a, b) respectively, simplify.

We have to note that not all possible magnetic fields $\boldsymbol{H}_0$, and therefore, not all vector fields $\boldsymbol{u}$ entering our equations (2.6g) or (2.9a), are permissible. Only such vector fields $\boldsymbol{H}_0$ which satisfy the magnetohydrostatic equation (1.14) of the undisturbed field are allowed.

Equation (1.14) written down for the vector field $\boldsymbol{u}$ has the form

$$\boldsymbol{u} \times \operatorname{curl} \boldsymbol{u} = -\operatorname{grad} p_0 \tag{3.1a}$$

or

$$\operatorname{curl}(\boldsymbol{u} \times \operatorname{curl} \boldsymbol{u}) = 0 . \tag{3.1b}$$

In the degenerated case of a force-free field we have instead of (3.1a), (3.1b) as the magnetohydrostatic equation

$$\boldsymbol{u} \times \operatorname{curl} \boldsymbol{u} = 0 . \tag{3.2}$$

We will consider the degenerated case of the force-free field separately. It is now possible to choose our hitherto, unspecified coordinate system in just such a way as to satisfy the equations (3.1a), (3.1b) [in the case of a force-free field (3.2)] automatically.

First we note that according to (3.1a) $\boldsymbol{u}$ and curl $\boldsymbol{u}$ are located in a surface. We choose, therefore, this surface as a coordinate surface. The coordinate line x^1 shall be directed along $\boldsymbol{u}$ and the coordinate line x^2 along curl $\boldsymbol{u}$. Magnetic field lines and electric current lines are, therefore, coordinate lines. The third coordinate line x^3 shall be directed along grad p_0 and is, therefore, perpendicular to x^1 and x^2.

The metric tensor has thus the form

$$g_{ik} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix} \tag{3.3a}$$

or

$$g_{13} = g_{23} = 0 . \tag{3.3b}$$

The contravariant components of $\boldsymbol{u}$ are

$$u^\alpha = \{u^1, 0, 0\} . \tag{3.4}$$

The vector curl $\boldsymbol{u}$ shall possess only a component along the direction of x^2. The contravariant components of curl $\boldsymbol{u}$ are given by

$$\operatorname{curl} u|^\alpha = g^{-1/2}\, \varepsilon^{\alpha\beta\gamma}\, u_{\gamma;\beta} \tag{3.5}$$

The requirement that the electric current shall possess no components along x^1 and x^3 results in the two equations following from (3.5):

$$\varepsilon^{1\beta\gamma}\, u_{\gamma;\beta} = 0 , \qquad \varepsilon^{3\beta\gamma}\, u_{\gamma;\beta} = 0$$

or

$$u_{3,2} - u_{2,3} = 0 , \tag{3.6a}$$

$$u_{2,1} - u_{1,2} = 0 . \tag{3.6b}$$

Next we can specify the third coordinate line x^3 in just such a way as to make

$$p_0 = A\, x^3, \qquad A = \text{const} . \tag{3.7}$$

The gradient of p_0 has then the following covariant components

$$\operatorname{grad} p_0\,|_\alpha = \{0, 0, A\} . \tag{3.8}$$

Or by virtue of (3.1a):

$$\boldsymbol{u} \times \operatorname{curl} \boldsymbol{u}\,|_\alpha = \{0, 0, -A\} . \tag{3.9}$$

Now we calculate

$$\begin{aligned} \boldsymbol{u} \times \operatorname{curl} \boldsymbol{u}\,|_\sigma &= \varepsilon^{\alpha\beta\gamma}\, \varepsilon_{\sigma\varrho\alpha}\, u^\varrho\, u_{\gamma;\beta} \\ &= (\delta_\sigma{}^\beta\, \delta_\varrho{}^\gamma - \delta_\varrho{}^\beta\, \delta_\sigma{}^\gamma)\, u^\varrho\, u_{\gamma;\beta} \\ &= u^\alpha (u_{\alpha,\sigma} - u_{\sigma,\alpha}) \end{aligned} \tag{3.10}$$

or remembering (3.4) we get:

$$\boldsymbol{u} \times \operatorname{curl} \boldsymbol{u}\,|_\sigma = u^1 (u_{1,\sigma} - u_{\sigma,1}) . \tag{3.11}$$

Combining (3.11) with (3.9) yields the three equations:

$$u_{1,1} - u_{1,1} = 0 \equiv 0 , \tag{3.12a}$$

$$u_{1,2} - u_{2,1} = 0 , \tag{3.12b}$$

$$u^1 (u_{1,3} - u_{3,1}) = -A . \tag{3.12c}$$

Of these equations (3.12a) is an identity and (3.12b) equivalent with (3.6a). We note furthermore, that because of (3.3a)

$$u_3 = g_{13}\, u^1 = 0$$

so that in summary:

$$u_{1,2} - u_{2,1} = 0 , \tag{3.13a}$$

$$u_{2,3} = 0 , \qquad u^1\, u_{1,3} = -A . \tag{3.13b, c}$$

It is now easy to check that the equation (3.1b) is then satisfied automatically.

This equation has the covariant form

$$\varepsilon^{\alpha\varrho\sigma} [u^1 (u_{1,\sigma} - u_{\sigma,1})]_{;\varrho} = 0 \tag{3.14}$$

and is satisfied for $\alpha = 1, 2, 3$ by virtue of (3.13 a, b, c). If $u_2 \neq 0$, then it is possible to give (3.13c) still another form which does not involve derivatives of $\boldsymbol{u}$.

It is
$$u_2 = g_{12}\, u^1 \tag{3.15}$$
therefore,
$$u_{2,3} = g_{12,3}\, u^1 + g_{12}\, u^1{}_{,3} = 0 \tag{3.16}$$
the latter by virtue of (3.13b).

Because of
$$u_1 = g_{11}\, u^1 \tag{3.17}$$
one has
$$u_{1,3} = g_{11,3}\, u^1 + g_{11}\, u^1{}_{,3}\,. \tag{3.18}$$
Eliminating $u^1{}_{,3}$ from (3.16) and (3.18) yields
$$u_{1,3} = g_{11}\, u^1 \left(\ln \frac{g_{11}}{g_{12}}\right)_{,3} = u_1 \left(\ln \frac{g_{11}}{g_{12}}\right)_{,3}\,. \tag{3.19}$$
From this and since, $u_1 u^1 = (u)^2$ it follows instead of (3.13c):
$$(u)^2 \left(\ln \frac{g_{12}}{g_{11}}\right)_{,3} = A\,. \tag{3.20}$$

We treat now the degenerated case of a force-free field defined by equation (3.2). Since in a force-free field the magnetic field lines and the electric current lines are parallel everywhere they do not more form a surface which could be used as a coordinate surface. We have, therefore, to proceed in a different way.

From equation (3.2) it follows that to each force-free field exists a scalar function $\Phi(x, y, z)$ depending in general on all three space coordinates and obeying the equation:
$$\operatorname{curl} \boldsymbol{u} = \boldsymbol{u}\, \Phi\,. \tag{3.21}$$
We apply the operation div on both sides and obtain by virtue of (1.26):
$$\boldsymbol{u} \cdot \operatorname{grad} \Phi = 0\,. \tag{3.22}$$
Equation (3.22) shows that $\boldsymbol{u}$ is located in the plane $\Phi = \text{const}$. It is therefore, suggestive to choose the plane $\Phi = \text{const.}$ as a coordinate surface. In this surface we introduce two coordinate lines x^1 and x^2. The x^1 line shall be directed along the magnetic field line. The x^2 line shall be located in the surface $\Phi = \text{const.}$, but otherwise be unspecified in its direction with regard to x^1. The x^3 line is chosen to be parallel to grad Φ. The metric tensor has, thus the same vanishing elements, as in the preceeding case given by (3.3a, b).

The contravariant components of $\boldsymbol{u}$ are also given by
$$u^\alpha = \{u^1, 0, 0\} \tag{3.23}$$
and here again we can put
$$\Phi = A\, x^3\,. \tag{3.24}$$
With (3.24) we obtain
$$\operatorname{grad} \Phi \,|_\alpha = \{0, 0, A\}\,. \tag{3.25}$$
The covariant form of equation (3.22)
$$u^\alpha\, \Phi_{,\alpha} = 0 \tag{3.26}$$
is then satisfied automatically.

In order to satisfy equation (3.2) we write down the covariant form of (3.2):
$$u^1 (u_{1,\sigma} - u_{\sigma,1}) = 0; \quad \sigma = 1, 2, 3 \tag{3.27}$$
which yields the two equations
$$u_{1,2} - u_{2,1} = 0\,, \quad u_{1,3} = 0\,. \tag{3.28a, b}$$
With equation (3.28a) and (3.28b) are then the second and third component of (3.21) automatically satisfied.

To obtain the first component of (3.21) we calculate
$$\operatorname{curl} \boldsymbol{u}\,|^1 = g^{-1/2}\, \varepsilon^{1\beta\gamma}\, u_{\gamma;\beta} = -\,(1/\sqrt{g})\; u_{2,3} \tag{3.29}$$
Now it is
$$u_2 = g_{12}\, u^1$$
therefore
$$u_{2,3} = g_{12,3}\, u^1 + g_{12}\, u^1{}_{,3} \tag{3.30}$$
and
$$u_1 = g_{11}\, u^1$$
therefore
$$u_{1,3} = g_{11,3}\, u^1 + g_{11}\, u^1{}_{,3} = 0 \tag{3.31}$$
the latter by virtue of (3.28b).

Eliminating $u^1{}_{,3}$ from (3.30) and (3.31) yields:
$$u_{2,3} = g_{12} \left(\ln \frac{g_{12}}{g_{11}}\right)_{,3} u^1\,. \tag{3.32}$$
Substituting this result into the first component of equation (3.21) results in:
$$A\, x^3 = \frac{g_{12}}{\sqrt{g}} \left(\ln \frac{g_{11}}{g_{12}}\right)_{,3} \tag{3.33}$$
therefore
$$A\, \sqrt{g}\, u^1\, x^3 = u_{2,3}\,. \tag{3.34}$$
From (3.33) follows that the x^2 line is, in general, not be directed perpendicularly to the x^1 line. $g_{12} = 0$ implies $A = 0$. $A = 0$, however, implies according to (3.21) and (3.25) that curl $\boldsymbol{u} = 0$, which means a current free field.

4. The Hydromagnetic Wave Equation in the Special Coordinate System

In writing out the wave equation in the special coordinate system we restrict ourselves to the simpler case of waves which obey the additional condition div $\boldsymbol{v}=0$.

It is also, however, easy to write down the wave equation for the more general case. We are especially interested to find solutions corresponding to ALFVÉN waves which just satisfy the condition div $\boldsymbol{v}=0$. We discuss the case of a force-free field here again later.

Our wave equation is given by (2.9a) with the constraint (2.9b). This wave equation in the special coordinate system can be brought into the form:

$$\varrho_0\,\omega^2\,v^\sigma = u^1\,g^{\sigma\varrho}[\{g_{\varrho 1}\,u^1{}_{,\beta}\,v^\beta - g_{\varrho\mu}\,u^1\,v^\mu{}_{,1}\}_{,1} - \{g_{11}\,u^1{}_{,\beta}\,v^\beta - g_{1\mu}\,u^1\,v^\mu{}_{,1}\}_{,\varrho}] + g^{\sigma\varrho}\,[\,(u_{\beta,\varrho}-u_{\varrho,\beta})\,(v^\beta{}_{,1}\,u^1 - v^\gamma\,u^\beta{}_{,\gamma}) + (v^\mu\,u^1\,(u_{1,\mu}-u_{\mu,1}))_{,\varrho}] \tag{4.1}$$

or written out in components [using (3.13a, b, c)] yields:

$$\varrho_0\,\omega^2\,v^1 = u^1\,g^{12}[\{g_{12}\,(u^1{}_{,1}\,v^1 + u^1{}_{,2}\,v^2 + u^1{}_{,3}\,v^3) - g_{12}\,u^1\,v^1{}_{,1} - g_{22}\,u^1\,v^2{}_{,1}\}_{,1} - \{g_{11}\,(u^1{}_{,1}\,v^1 + u^1{}_{,2}\,v^2 + u^1{}_{,3}\,v^3) - g_{11}\,u^1\,v^1{}_{,1} - g_{12}\,u^1\,v^2{}_{,1}\}_{,2}] - A\,g^{12}\,v^3{}_{,2}\,, \tag{4.2a}$$

$$\varrho_0\,\omega^2\,v^2 = u^1\,g^{22}[\{g_{12}\,(u^1{}_{,1}\,v^1 + u^1{}_{,2}\,v^2 + u^1{}_{,3}\,v^3) - g_{12}\,u^1\,v^1{}_{,1} - g_{22}\,u^1\,v^2{}_{,1}\}_{,1} - \{g_{11}\,(u^1{}_{,1}\,v^1 + u^1{}_{,2}\,v^2 + u^1{}_{,3}\,v^3) - g_{11}\,u^1\,v^1{}_{,1} - g_{12}\,u^1\,v^2{}_{,1}\}_{,2}] - A\,g^{22}\,v^3{}_{,2}\,, \tag{4.2b}$$

$$\varrho_0\,\omega^2\,v^3 = -\,u^1\,g^{33}[\,(g_{33}\,u^1\,v^3{}_{,1})_{,1} + \{g_{11}\,(u^1{}_{,1}\,v^1 + u^1{}_{,2}\,v^2 + u^1{}_{,3}\,v^3) - g_{11}\,u^1\,v^1{}_{,1} - g_{12}\,u^1\,v^2{}_{,1}\}_{,3}] - A\,g^{33}\,(v^3{}_{,1} + v^3{}_{,3}) + \frac{A\,g^{33}}{u^1}\,(v^1\,u^1{}_{,1} + v^2\,u^1{}_{,2} + v^3\,u^1{}_{,3})\,. \tag{4.2c}$$

This must be supplemented by
$$(\sqrt{g}\,v^1)_{,1} + (\sqrt{g}\,v^2)_{,2} + (\sqrt{g}\,v^3)_{,3} = 0\,. \tag{4.3}$$

It is now easy to see that this system of partial differential equations, (4.2a, b, c), together with (4.3) can be reduced to the problem of solving an ordinary differential equation if the following conditions are satisfied:

$$v^1 = v^3 = 0\,,\qquad g_{12} = g^{12} = 0\,,\qquad \partial/\partial x^2 = 0\,. \tag{4.4a, b, c}$$

Equation (4.2a), (4.2c) and (4.3) are then satisfied identically. For equation (4.2b) we obtain

$$\varrho_0\,\omega^2\,v^2 = -\,u^1\,g^{22}\,(g_{22}\,u^1\,v^2{}_{,1})_{,1}\,. \tag{4.5}$$

Condition (4.4b) is satisfied if the current is directed perpendicularly to the magnetic field line. Condition (4.4b), together with (4.4c), implies furthermore as shown in the appendix that the x^2 line is directed along the binormal of the x^1 line and the x^1 line has no torsion.

Equation (4.5) describes an ALFVÉN wave with a varying refractive index and can be treated for instance by the W.K.B. method. If the field is homogeneous at $x^1 = -\infty + \infty$, it can also be treated by BORN approximation.

It is easy to bring equation (4.5) to a form describing a wave propagating in a medium of variable refractive index.

This form is given by

$$\left\{\frac{\partial^2}{\partial x^2\,\partial x^2} + \frac{\varrho_0\,\omega^2}{(u^1)^2} - \frac{1}{4}\,[\,(\ln g_{22}\,u^1)_{,1}]^2 - \frac{1}{2}\,(\ln g_{22}\,u^1)_{,11}\right\}[\,(g_{22}\,u^1)^{1/2}\,v^2] = 0\,. \tag{4.6}$$

We are turning finally to the case of a force-free field. In the special, for the force-free field investigated coordinate system, the form of the wave equation is the same as equation (4.1).

Writing this out in components but taking now into account (3.28a, b) and (3.34) yields:

$$\varrho_0\,\omega^2\,v^1 = u^1\,g^{12}[\{g_{12}\,(u^1{}_{,1}\,v^1 + u^1{}_{,2}\,v^2 + u^1{}_{,3}\,v^3) - g_{12}\,u^1\,v^1{}_{,1} - g_{22}\,u^1\,v^2{}_{,1}\}_{,1} - \{g_{11}\,(u^1{}_{,1}\,v^1 + u^1{}_{,2}\,v^2 + u^1{}_{,3}\,v^3) - g_{11}\,u^1\,v^1{}_{,1} - g_{12}\,u^1\,v^2{}_{,1}\}_{,2}] + A\,g^{12}\,\sqrt{g}\,x^3\,(u^1)^2\,v^3{}_{,1}\,, \tag{4.7a}$$

$$\varrho_0\,\omega^2\,v^2 = u^1\,g^{22}[\{g_{12}\,(u^1{}_{,1}\,v^1 + u^1{}_{,2}\,v^2 + u^1{}_{,3}\,v^3) - g_{12}\,u^1\,v^1{}_{,1} - g_{22}\,u^1\,v^2{}_{,1}\}_{,1} - \{g_{11}\,(u^1{}_{,1}\,v^1 + u^1{}_{,2}\,v^2 + u^1{}_{,3}\,v^3) - g_{11}\,u^1\,v^1{}_{,1} - g_{12}\,u^1\,u^2{}_{,1}\}_{,2}] + A\,g^{22}\,\sqrt{g}\,x^3\,(u^1)^2\,v^3{}_{,1}\,, \tag{4.7b}$$

$$\varrho_0\,\omega^2\,v^3 = -\,u^1\,g^{33}[\,(g_{33}\,u^1\,v^3{}_{,1})_{,1} + \{g_{11}\,(u^1{}_{,1}\,v^1 + u^1{}_{,2}\,v^2 + u^1{}_{,3}\,v^3) - g_{11}\,u^1\,v^1{}_{,1} - g_{12}\,u^1\,v^2{}_{,1}\}_{,3}] - A\,g^{33}\,\sqrt{g}\,x^3\,(u^1)^2\,v^2{}_{,1}\,. \tag{4.7c}$$

These equations must be supplemented by equation (4.3).

Here again it is possible to reduce the solution of this system of partial differential equations to the solution of an ordinary differential equation if

$$v^1 = v^3 = 0\,, \tag{4.8a}$$

$$g_{12} = g^{12} = 0\,, \tag{4.8b}$$

$$\partial/\partial x^2 = 0\,. \tag{4.8c}$$

From (4.8b) it follows then by virtue of (3.33)

$$A = 0 \tag{4.9}$$

and this is according to (3.21) together with (3.24) the case of a current-free field. The resulting single wave equation is given by

$$\varrho_0\,\omega^2\,v^2 = -u^1\,g^{22}(g_{22}\,u^1\,v^2{}_{,1})_{,1} \tag{4.10}$$

which is of the same form as equation (4.5).

5. Conclusion

It has been shown that hydromagnetic wave equations can be considerably simplified by going to certain curvilinear coordinate systems intrinsically adapted to the shape and differential geometric property of the magnetic field.

In simple cases as the one in which the current lines are perpendicular to the field lines or when the current is zero the solution of the set of partial differential equations describing Alfvén waves can be reduced to the solution of an ordinary differential equation.

Appendix

Torsion and Curvature of the Field Lines

$\boldsymbol{t}$, $\boldsymbol{n}$, and $\boldsymbol{b}$ be the tangent, normal and binormal vector of the x^1 line, normalized to one:

$$t_\alpha\,t^\alpha = n_\alpha\,n^\alpha = b_\alpha\,b^\alpha = 1\,. \tag{A.1}$$

For the following we assume that

$$g_{12} = g_{13} = g_{23} = 0, \qquad \partial/\partial x^2 = 0 \tag{A.2}$$

by which the set of partial differential equations could be reduced.

For the components of the tangent vector we have

$$t^\mu = \{(g_{11})^{-1/2}, 0, 0\} = \{(g^{11})^{1/2}, 0, 0\}\,. \tag{A.3}$$

The only nonvanishing components of the normal and binormal vector which have to be calculated are

$$n^\mu = \{0,\ n^2,\ n^3\}\,, \tag{A.4}$$

$$b^\mu = \{0,\ b^2,\ b^3\}\,. \tag{A.5}$$

The Frenet formulas are (in covariant form):

$$t^\mu{}_{;\nu}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = \varkappa\,n^\mu\,, \tag{A.6}$$

$$b^\mu{}_{;\nu}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = \tau\,n^\mu\,. \tag{A.7}$$

Further since we like to calculate n^μ and b^μ for the x^1 line.

$$\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = \{(g_{11})^{-1/2}, 0, 0\} = \{(g^{11})^{1/2}, 0, 0\}\,. \tag{A.8}$$

From (A.6) we obtain

$$n^\mu = \varkappa^{-1}\,t^\mu{}_{;1}(g^{11})^{1/2} = \varkappa^{-1}(g^{11})^{1/2}(t^\mu{}_{,1} + \Gamma^\mu{}_{11}\,t^1)\,. \tag{A.9}$$

From this and observing (A.2) we get

$$n^1 = 0, \qquad n^2 = 0,$$

$$n^3 = -\frac{1}{2}\,\varkappa^{-1}\,\frac{g_{11,3}}{g_{11}\,g_{33}}\,. \tag{A.10}$$

Contracting $\quad g_{33}\,n^3\,n^3 = 1 \quad$ (A.11)

gives the value for the curvature $\varkappa$:

$$\varkappa = \frac{1}{2}\,\frac{g_{11,3}}{g_{11}\,\sqrt{g_{33}}}\,. \tag{A.12}$$

The components of the binormal vector can now be calculated by the formula

$$b^\mu = g^{-1/2}\,\varepsilon^{\mu\alpha\beta}\,t_\alpha\,n_\beta\,. \tag{A.13}$$

From this follows

$$b^1 = 0, \qquad b^3 = 0,$$

$$b^2 = \tfrac{1}{2}\,\varkappa^{-1}\,g^{-1/2}(g^{11})^{1/2}\,g_{11,3}\,. \tag{A.14}$$

(A.14) shows that the binormal is directed along the x^2 line.

The torsion finally can be calculated from (A.7):

$$\begin{aligned}\tau &= -b^\mu{}_{;\nu}\,n_\mu\,\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\\ &= -(g^{11})^{1/2}(b^\mu{}_{,1} + \Gamma^\mu_{1\alpha}\,b^\alpha)\,n_\mu \qquad \text{(A.15)}\\ &= -(g^{11})^{1/2}\,g_{33}\,\Gamma_{21}{}^3\,b^2\,n^3 = 0\,.\end{aligned}$$

The latter since in an orthogonal coordinate system

$$\Gamma_{21}{}^3 = 0\,.$$

The torsion of the x^1 line is, therefore, zero.